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# ON GENERAL CONSERVATIVE END LOADING OF PRETWISTED RODS

MEHDI FARSHAD†

School of Engineering, Pahlavi University, Shiraz, Iran

**Abstract**—Treated is a pretwisted rod subjected to combined action of generally nonconservative force and couple applied at the end. The general conservative end loads are obtained through the consideration of an adjoint field and from the study of the bilinear concomitant of the equations of motion. Several special cases of interest are discussed.

## 1. INTRODUCTION

THE THEORY of elastic stability of systems subjected to nonconservative loads has received considerable attention. Two of the salient features of nonconservative stability problems which have been brought into light by various investigators are that the applied forces in these problems are not derivable from a potential, and that any fruitful stability analysis of these problems has to be carried out through the kinetic approach. From the mathematical point of view it has been established that the notion of nonconservative system is closely linked to the non-self adjointness of the operators governing the behaviour of the system.

Along with the investigations related to the nonconservative loadings there has been a continuing interest in the studies of conservative systems. Various special cases of conservative loadings applied to particular elastic bodies has been treated in the literature in connection with allied problems. Some of these works utilize the concept of self-adjointness of the system while others base the derivation of conservative loading system on the potential function. References [1–16] contain some of the investigations, related to the present work, which had been carried out in this area.

The aim of this paper is to obtain and discuss the most general conservative end loadings to which a pretwisted rod may be subjected. The choice of a pretwisted rod problem for this investigation is due to its technical importance and also its generality in embodying some other slender body problems as special cases. In the present study to the end of determination of conservative loading conditions the notion of self adjointness and/or non-self adjointness of the operators governing the system is fully utilized. The general conservative boundary conditions of a pretwisted rod subjected to combined action of axial force and torsional couple is obtained and some special cases of importance are discussed.

† Associate Professor, Department of Civil Engineering.

## 2. GOVERNING EQUATIONS OF PRETWISTED RODS

### 1. *Kinematical considerations*

Consider a rod whose centerline in the undeformed state is a spatial curve. To carry out the subsequent analyses we define the following sets of coordinate systems in this undeformed state.

(a) Global coordinate system, $X_0Y_0Z_0$, at some fixed point of the rod.

(b) A local coordinate system, $xyz$, at the center of mass of any cross-section of the bar. Two axes of this set are in the cross-section and are principal axes and the third axis is taken along the tangent to the centerline curve.

(c) The third system of coordinates, $abc$, is obtained by rotation of the previous set about a horizontal line so that an axis parallel to $Z$ axis is obtained.

For a spatial curve the rate of charge of orientation of $xyz$ relative to $X_0Y_0Z_0$ is determined by the knowledge of three parameters $p_0$, $q_0$, $r_0$, where $p_0$, $q_0$ are the components of curvature along $x$,$y$ and $r_0$ is the rate of twist of the curve.

If $S$ is the arc length of the bar in the undeformed state, then the vector representing the rate of change of orientation of $xyz$ relative to $X_0Y_0Z_0$ can be written as

$$\boldsymbol{\omega} = p_0\mathbf{i}+q_0\mathbf{j}+r_0\mathbf{k} \tag{1}$$

in which $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the unit vectors along the $x$, $y$, $z$ axes, respectively.

If the warping of the cross-sections of the bar are neglected, and the small deformation of the slender rod under consideration is limited to stretching, bending and torsion, then the following two vectors completely describe the six degrees of freedom of deformation.

$$\mathbf{U}(s) = u\mathbf{i}+v\mathbf{j}+w\mathbf{k} \tag{2}$$

$$\mathbf{H}(s) = \theta_x\mathbf{i}+\theta_y\mathbf{j}+\theta_z\mathbf{k}. \tag{3}$$

### 2. *Dynamical equations*

If the resultant force vector and couple vector in a bar cross-section (referred to deformed state) are denoted by $\mathbf{Q}$ and $\mathbf{M}$, respectively, the force equations of motion of a bar element can be written as follows (see for instance, [13], [17], [18]).

$$\mathbf{Q}'+\boldsymbol{\omega}\times\mathbf{Q}+\mathbf{f} = 0 \tag{4}$$

$$\mathbf{M}'+\boldsymbol{\omega}\times\mathbf{M}+\mathbf{k}\times\mathbf{Q}+\mathbf{m} = 0 \tag{5}$$

where prime designates partial derivative with respect to $S$ in the local system $xyz$. $\mathbf{f}$ and $\mathbf{m}$ are force and couple (inertia and/or external) applied to the bar element.

In the subsequent analyses we shall disregard the effects of shear deformation and rotatory inertia. Furthermore, we shall restrict the ensuing treatments to the case of a naturally straight and pretwisted rod. Mathematically the latter consideration leads to setting $p_0$ and $q_0$ equal to zero.

With the above assumptions one can readily observe that in equations (4) and (5) the longitudinal and torsional motions of the bar decouple from the other type of deformations. Two equations yield the axial force and torsional couple in the rod. If the corresponding axial and torsional inertia and applied forces are neglected, then the result of such a separate study is a constant axial force $P$ and a constant torsional couple $L$ at any cross-section of the bar.

3. *Constitutive relations*

Let **I** be a diadic defined by:

$$\mathbf{I} = \alpha\mathbf{ii}+\beta\mathbf{jj}+\gamma\mathbf{kk} \tag{6}$$

in which $\alpha$, $\beta$, are the bending rigidities about $x$, $y$ axes and $\gamma$ is the torsional rigidity.

We can write the constitutive relation of the elastic rod in the form

$$\mathbf{M} = \mathbf{I}\,.\,\mathbf{X} \tag{7}$$

where

$$\mathbf{X} = \mathbf{H}'+\boldsymbol{\omega}\times\mathbf{H} \tag{8}$$

so

$$M_x = \alpha\mathbf{X}_x, \qquad M_y = \beta\mathbf{X}_y, \qquad M_z = \gamma\mathbf{X}_z. \tag{9}$$

With the help of assumptions of zero shear deformation, and noting that torsional deformation may be treated as a separate problem the constitutive equations of the initially straight, pretwisted (constant rate of pretwist) bar under consideration are, in expanded form, as follows:

$$M_x = -\alpha(v''+2r_0u'-r_0^2v) \tag{10}$$

$$M_y = \beta(u''-2r_0v'-r_0^2u). \tag{11}$$

4. *Displacement equations of motion*

The displacement equations of lateral motion of an initially straight, pretwisted rod subjected to axial force and torsional couple are obtained by proper combination of equations in parts 1, 2 and 3. For stability analysis of this bar according to kinetic method we shall be concerned with the study of vibrations of the system in the vicinity of state whose stability is being investigated. To this end we let,

$$f_x = -m\frac{\partial^2 u}{\partial t^2}, \qquad f_y = -m\frac{\partial^2 v}{\partial t^2} \tag{12}$$

in which $m$ is the mass per unit length of the bar and $t$ is the time parameter. If we combine equations (4), (5), (10), (11) and (12) we obtain the following equations:

$$L_1(u,v) = \beta u'''' -(2\alpha r_0+2\beta r_0-L)v'''-(4\alpha r_0^2+2\beta r_0^2-3Lr_0-P)u''$$
$$+(2\alpha r_0^3+2\beta r_0^3-3Lr_0^2-2Pr_0)v'+(\beta r_0^4-Lr_0^3-Pr_0^2)u = -m\frac{\partial^2 u}{\partial t^2} \tag{13}$$

$$L_2(u,v) = \alpha v'''' +(2\alpha r_0+2\beta r_0-L)u'''-(2\alpha r_0^2+4\beta r_0^2-3Lr_0-P)v''$$
$$-(2\alpha r_0^3-2\beta r_0^3-3Lr_0^2-2Pr_0)u'+(\alpha r_0^4-Lr_0^3-Pr_0^2)v = -m\frac{\partial^2 v}{\partial t^2}. \tag{14}$$

We note that these equations contain the equations of Refs. [13] and [18] as special cases.

## 3. STABILITY CONSIDERATIONS BY KINETIC APPROACH

The pretwisted rod studied in this paper is assumed to be subjected to the most general end loading. This generalization of end-loading conditions allows the consideration of all types of loadings some of which may very well be nonconservative. Hence, the stability analysis of such a system calls for the kinetic approach which is based on the investigation of motion of the system about some state of deformation whose stability is under consideration [13].

It is shown [19] that for the stability analysis of a prismatic bar under general end loading one has to consider an adjoint system. In the present paper that notion is generalized to the treatment of stability of a pretwisted rod under the application of axial force and torsional moment.

The most general boundary conditions for the rod under study may be written in the form,

$$a_{ij}\zeta_j = 0 \qquad \begin{array}{l} j = 1, \ldots, 16 \\ i = 1, \ldots, 8 \end{array} \tag{15}$$

where $\zeta_j$ are the displacements, slopes, shear forces, and bending moments at both ends of the rod. Relations (15) are linearly independent and they can be solved for force quantities in terms of end displacements and end slopes.

With the differential equations of motion and boundary conditions completely specified we can write the following

$$\Delta = \int_0^1 [\bar{u}L_1(u, v) + \bar{v}L_2(u, v) - uL_1(\bar{u}, \bar{v}) - vL_2(\bar{u}, \bar{v})]\, d\xi. \tag{16}$$

$\xi$ is the dimensionless length coordinate; where a bar over the letter designates its corresponding adjoint function.

If we substitute operators $L_1$ and $L_2$ from (13) and (14) into (16) and perform necessary integration by parts, and noting that

$$Q_x = -\beta u''' + (2\beta r_0 + \alpha r_0 - L)v'' + (\beta r_0^2 - 2\alpha r_0^2 - 2Lr_0)u' - (\alpha r_0^3 - Lr_0^2)v \tag{17}$$

$$Q_y = -\alpha v''' - (2\alpha r_0 + \beta r_0 - L)u'' + (\alpha r_0^2 + 2\beta r_0^2 - 2Lr_0)v' - (\beta r_0^3 - Lr_0^2)u \tag{18}$$

we obtain

$$\begin{aligned} \Delta = \{&[-Q_x + r_0 M_x + (Lr_0 + P)u' - (Lr_0^2 + 2Pr_0)v]\bar{u} \\ &+ [-Q_y + r_0 M_y + (Lr_0 + P)v' + (Lr_0^2 + 2Pr_0)u]\bar{v} \\ &+ [-M_y - Lv' - (Lr_0 + P)u]\bar{u}' + [M_x + Lu' - (Lr_0 + P)v]\bar{v}' \\ &+ (\bar{Q}_x - r_0\bar{M}_x)u + (\bar{Q}_y - r_0\bar{M}_y)v + \bar{M}_y u' - \bar{M}_x v'\}_0^1 \end{aligned} \tag{19}$$

It was mentioned in the earlier part of the paper that $Q_x$, $Q_y$, $M_x$, $M_y$ are quantities referred to local coordinates of bar in deformed state (which is the same as $xyz$ system for small deformation). If $Q_a$, $Q_b$, $M_a$, $M_b$ are the corresponding quantities in $abc$ system they are given by the following relationships:

$$\begin{aligned} Q_a &= Q_x - P(u' - r_0 v) \\ Q_b &= Q_y - P(v' + r_0 u) \\ M_a &= M_x + L(u' - r_0 v) \\ M_b &= M_y + L(v' + r_0 u). \end{aligned} \tag{20}$$

## 4. ADJOINT FIELD

The system of equations (13) and (14) together with the boundary conditions (15) constitute the equations governing the problem of dynamic stability of a pretwisted bar subjected to the most general end loading. Let's consider an adjoint field $\bar{u}$, $\bar{v}$ which satisfies the same equations of motion and the following boundary conditions:

$$b_{ij}\bar{\zeta}_j = 0 \qquad \begin{matrix} j = 1, \ldots, 16 \\ i = 1, \ldots, 8 \end{matrix} \tag{21}$$

where $\bar{\zeta}_j$ are the displacements, slopes, shear forces and bending moments, associated with the adjoint field, at the ends of the bar.

## 5. GENERAL CONSERVATIVE END-LOADING

In order that the combination of the main field (governed by equations (13–15) and the adjoint field (governed by equations 13, 14 and 21) yield a conservative system we demand that $\Delta = 0$. This would result some relations among the constants $a_{ij}$ and $b_{ij}$ appearing in equations (15) and (21).

In this paper we shall restrict ourselves to the physical problem of a pretwisted rod one end of which is fixed. For such a problem conditions (15) and (21), if solved for force quantities, become

$$\left.\begin{aligned} Q_x &= \alpha_1 u + \alpha_2 v + \alpha_3 u' + \alpha_4 v' \\ Q_y &= \alpha_5 u + \alpha_6 + \alpha_7 u' + \alpha_8 v' \\ M_y &= \alpha_9 u + \alpha_{10} v + \alpha_{11} u' + \alpha_{12} v' \\ M_x &= \alpha_{13} u + \alpha_{14} v + \alpha_{15} u' + \alpha_{16} v' \end{aligned}\right\} \quad \text{at } \xi = 1. \tag{22}$$

and

$$\left.\begin{aligned} \bar{Q}_x &= \beta_1 \bar{u} + \beta_2 \bar{v} + \beta_3 \bar{u}' + \beta_4 \bar{v}' \\ \bar{Q}_y &= \beta_5 \bar{u} + \beta_6 \bar{v} + \beta_7 \bar{u}' + \beta_8 \bar{v}' \\ \bar{M}_y &= \beta_9 \bar{u} + \beta_{10} \bar{v} + \beta_{11} \bar{u}' + \beta_{12} \bar{v}' \\ \bar{M}_x &= \beta_{13} \bar{u} + \beta_{14} \bar{v} + \beta_{15} \bar{u}' + \beta_{16} \bar{v}'. \end{aligned}\right\} \quad \text{at } \xi = 1. \tag{23}$$

If the boundary conditions (22) and (23) are substituted into equations (19), an expression in terms of $u$, $v$, $u'$, $v'$ (at $\xi = 1$) and their adjoints is obtained. Treating these quantities as independent and, in general case, nonzero in order to have $\Delta = 0$ we require that the following relations hold:

$$\begin{aligned} \beta_1 &= \alpha_1 - \alpha_4 - \alpha_{13} r_0 + \alpha_{16} r_0 \\ \beta_2 &= \alpha_5 - \alpha_8 r_0 - \alpha_9 r_0 + \alpha_{12} r_0^2 - P r_0 \\ \beta_3 &= \alpha_9 - \alpha_{12} r_0 + P \\ \beta_4 &= -\alpha_{13} + \alpha_{16} r_0 \\ \beta_5 &= \alpha_2 + \alpha_3 r_0 - \alpha_{14} r_0 - \alpha_{15} r_0^2 + P r_0 \\ \beta_6 &= \alpha_6 + \alpha_7 r_0 - \alpha_{10} r_0 - \alpha_{11} r_0^2 \\ \beta_7 &= \alpha_{10} + \alpha_{11} r_0 \\ \beta_8 &= -\alpha_{14} - \alpha_{15} r_0 + P \end{aligned} \tag{24}$$

$$
\begin{aligned}
\beta_9 &= \alpha_3 - \alpha_{15} r_0 - L r_0 - P \\
\beta_{10} &= \alpha_7 - \alpha_{11} r_0 \\
\beta_{11} &= \alpha_{11} \\
\beta_{12} &= -\alpha_{15} - L \\
\beta_{13} &= -\alpha_4 + \alpha_{16} r_0 \\
\beta_{14} &= -\alpha_8 + \alpha_{12} r_0 + L r_0 + P \\
\beta_{15} &= -\alpha_{12} - L \\
\beta_{16} &= \alpha_{16}.
\end{aligned} \tag{24}
$$

With the above relations among the constants in boundary conditions, the conservatism of the combined system, consisting of the main field and the adjoint field, is assured. If we further require that the system be self-adjoint then we should write:

$$\beta_i = \alpha_i \qquad i = 1, \ldots, 16. \tag{25}$$

Hence for a self-adjoint system which from the physical point of view implies a conservative problem the relations (24) become

$$
\begin{aligned}
\alpha_2 - \alpha_5 + \alpha_9 r_0 - \alpha_{14} r_0 + L r_0^2 + 2 P r_0 &= 0 \\
\alpha_3 - \alpha_9 - \alpha_{15} r_0 - L r_0 - P &= 0 \qquad (26) \\
\alpha_4 + \alpha_{13} - \alpha_{16} r_0 &= 0 \\
\alpha_7 - \alpha_{10} - \alpha_{11} r_0 &= 0 \\
\alpha_8 + \alpha_{14} - \alpha_{12} r_0 - L r_0 - P &= 0 \qquad (26) \\
\alpha_{12} + \alpha_{15} + L &= 0
\end{aligned}
$$

these relations show that the most general conservative boundary conditions of an initially straight and pretwisted rod subjected to combined action of end force and couple (and with one end fixed) depend on ten parameters. In the following we shall consider some physically interesting cases as applications of the results obtained in equations (24) and (26).

(a) *Adjoint field corresponding to the follower force and follower couple field*

If the main field consists of a pretwisted bar subjected to the combined action of a follower force and follower couple (a nonconservative problem) then $\alpha_i$ are all zero and we have

$$Q_x = Q_y = M_x = M_y = 0 \quad \text{at } \xi = 1 \tag{27}$$

according to (24) an adjoint field which when combined with this nonconservative field will result a conservative state of loading is given by

$$
\left.
\begin{aligned}
\bar{Q}_x &= P(\bar{u}' - r_0 \bar{v}) \\
\bar{Q}_y &= P(\bar{u}' + r_0 u)
\end{aligned}
\right\} \quad \text{at } \xi = 1 \tag{28}
$$

$$
\left.
\begin{aligned}
\bar{M}_y &= P\bar{v} - L(\bar{u}' - r_0 v) \\
\bar{M}_y &= -P\bar{u} - L(\bar{v}' + r_0 u)
\end{aligned}
\right\} \quad \text{at } \xi = 1 \tag{28}
$$

as subcases one may set combinations of $P$, $L$ or $r_0$ equal to zero, each of which would pertain to a physical problem. In particular case if we let $r_0 = 0$ the results of Refs. [20] and [21] is reproduced.

(b) *Prismatic bar subjected to compressive force and torque*

If we set $r_0 = 0$ we will obtain the governing equations of a prismatic bar which is subjected to compressive force and torsional moment. The governing equations are obtained from (13) and (14) as,

$$\alpha u'''' + Lv''' + Pu'' + m\frac{\partial^2 v}{\partial t^2} = 0 \tag{29}$$

$$\beta v'''' - Lu''' + Pv'' + m\frac{\partial^2 v}{\partial t^2} = 0. \tag{30}$$

This problem governed by the above equations together with some end conditions leading to a nonconservative loading and also some end conditions constituting conservative cases is studied before [13]. Here we shall obtain the general boundary conditions which make this system a conservative stability problem. To that end we let $r_0 = 0$ in (22), so we obtain:

$$\left.\begin{aligned} Q_x &= \alpha_1 u + \alpha_2 v + \alpha_3 u' + \alpha_4 v' \\ Q_y &= \alpha_2 u + \alpha_5 u + \alpha_7 u' + \alpha_8 v' \end{aligned}\right\} \text{ at } \xi = 1 \tag{31}$$

$$\left.\begin{aligned} M_y &= (\alpha_3 - P)u + \alpha_7 v + \alpha_{11} u' + \alpha_{12} v' \\ M_x &= -\alpha_4 u - (\alpha_8 + P)v - (\alpha_{12} + L)u' + \alpha_{16} v' \end{aligned}\right\} \text{ at } \xi = 1. \tag{31}$$

We note that conservative problems discussed in [13] are special cases of the above results.

Some specific conservative cases of interest may be extracted from (31) by setting all or some of the independent parameters equal to zero.

Consider the case of a follower load. This case is realized by setting $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$, $\alpha_4 = 0$, $\alpha_6 = 0$, $\alpha_7 = 0$, $\alpha_8 = 0$. So we get

$$\left.\begin{aligned} Q_x &= 0 \\ Q_y &= 0 \\ M_y &= -Pu + \alpha_{11} u' + \alpha_{12} v' \\ M_x &= -Pv - (\alpha_{12} + L)u' + \alpha_{16} v' \end{aligned}\right\} \text{ at } \xi = 1 \tag{32}$$

relations (32) show that it is possible to get a conservative problem in a column subjected to a follower load.

As another case we consider the problem of follower torsional couple. Equations (31) become,

$$\left.\begin{aligned} Q_x &= \alpha_1 u + \alpha_2 v + Pu' \\ Q_y &= \alpha_2 u + \alpha_6 v - Pv' \\ M_y &= 0 \\ M_x &= 0. \end{aligned}\right\} \text{ at } \xi = 1 \tag{33}$$

(c) *Subtangential and supertangential loadings*

Consider the case in which the axial force and torsional couple make an angle with the deformed centerline of the rod. For this case we have

$$\left.\begin{aligned} Q_x &= kP(u'-r_0v) \\ Q_y &= lP(v'+r_0u) \\ M_y &= mL(v'+r_0u) \\ M_x &= nL(u'-r_0u). \end{aligned}\right\} \quad \text{at } \xi = 1 \tag{34}$$

The limiting cases of follower loading and constant direction loadings are obtained by assigning appropriate values to the parameters $k$, $l$, $m$, $n$. With the help of conditions (26), we obtain

$$k = 1, \qquad l = 1, \qquad m+n-1 = 0. \tag{35}$$

The above relations when employed in (34) result a state of loading in which the force $P$ remains along $Z$ axes and the sub (super) tangentiality of torsional couple is such that the third of (35) holds. A special subcase is that of $P = 0$ with similar conclusions.

(d) *Follower type conservative end loadings*

If the axial end force is taken to be along the deformed centerline of the rod, then the conditions (26) show that for conservatism of the system, and for nonzero $P$, we must have $r_0 = 0$. Satisfying this requirement, we obtain:

$$\left.\begin{aligned} Q_x &= 0 \\ Q_y &= 0 \\ M_y &= -Pu+\alpha_{11}u'+\alpha_{12}v' \\ M_x &= Pv-(\alpha_{12}+L)u'+\alpha_{16}v'. \end{aligned}\right\} \quad \text{at } \xi = 1 \tag{36}$$

As the second case of conservative follower loading we consider the example of a rod subjected at the end to an axial force and a follower torsional couple. A conservative loading corresponding to this state is realizable only if $L = 0$. Letting $L = 0$ we get

$$\left.\begin{aligned} Q_x &= \alpha_1u+\alpha_2v+Pu' \\ Q_y &= (\alpha_2+2Pr_0)u+\alpha_6v+Pv' \\ M_y &= 0 \\ M_x &= 0. \end{aligned}\right\} \quad \text{at } \xi = 1 \tag{37}$$

As a subcase we may set $P = 0$.

Some examples of conservative loadings are shown in Figs. 1–4.

## 6. CONCLUDING REMARKS

In this paper the problem of a pretwisted bar subjected to general conservative state of end loading was treated. The state of loading in general case is composed of forces and

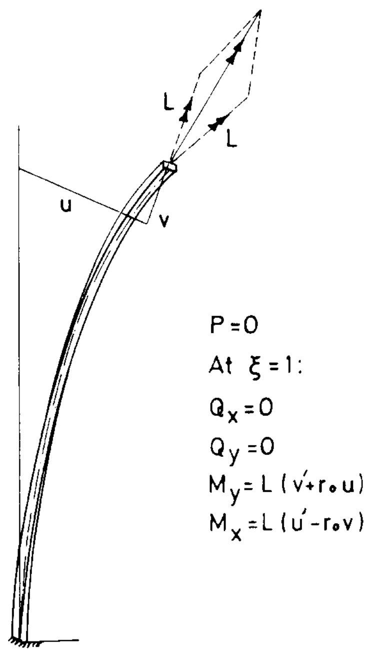


FIG. 1. Subtangential end couple. (An example of cases a and b).

couples. Utilizing the concepts of kinetic approach of stability analysis together with the notion of adjoint field we obtain the appropriate combination of end loading which makes the problem conservative and the whole system self-adjoint. The specific cases extracted from the general results depict some interesting features; namely, it may be possible to

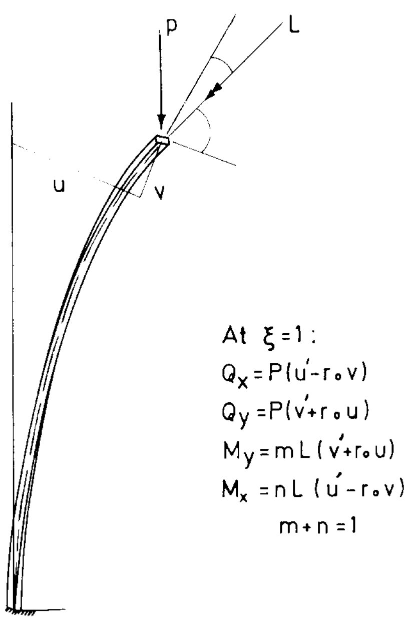


FIG. 2. Subtangential compressive loading. (An example of cases b and c).

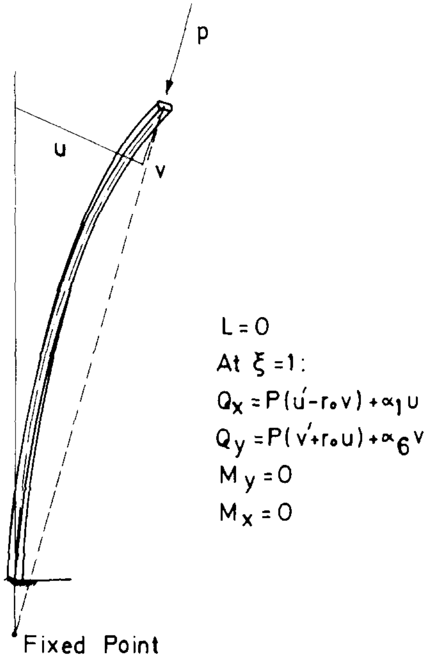


FIG. 3. Conservative polar compressive load.

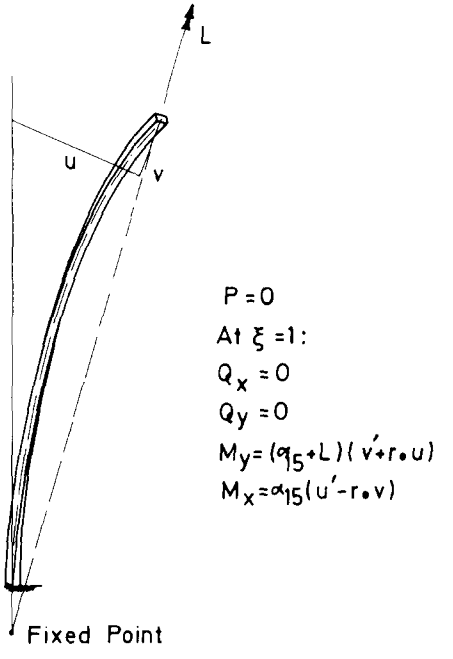


FIG. 4. Conservative polar torque.

combine a nonconservative force system with another system such that the result would be conservative. Also, in one of the specific cases the conditions of subtangentiality of a torsional couple applied at the bar end in a conservative fashion is obtained, i.e. between two limiting nonconservative cases of axial and follower torsional couple a conservative state is found. The approach used in studying the present problem appears to be quite general and it seems fruitful to establish the conservative state of loadings for other systems by this systematic method.

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**Абстракт**—Рассматривается предварительно закрученный стержень, подверженный совместному действию обще не-консервативного усилия и момента, приложенных на конце. Общие консерватные нагрузки на конце получаются путем исследования сопряженного поля и билинейного сопуствования уравнений движения. Обсуждаютя некоторые, проявляющие интерес, случаи.